Name: Mr. Solutions **RCS ID:** @rpi.edu

1. We'll consider the *center* of a graph to be the induced subgraph of all vertices of minimum eccentricity, or all vertices u where the greatest distance $d(u, v)$ to other vertices v is minimum. Use induction to prove that the *center* of a tree is either K_1 or K_2. Hint: consider what vertices might be the greatest distance from any center. (15 pts)

We'll do induction on vertices

Base:  obviously their centers are just K_1 or K_2

I.H.: Suppose for some $P(k)$ that is a tree its center is K_1 or K_2

I.S.: Consider all leaves in some tree $P(n)$ $n > k$. Removing these leaves will decrement all maximum shortest paths from the center by exactly one. We invoke our I.H. on our $P(k)$ case. We now consider re-adding our leaves to return to $P(n)$. We note that all maximum shortest paths from the center will increase by 1 and all general maximum shortest paths will aswell. Hence, the center is unchaged and therefore K_1 or K_2 $\square$

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2. We use the transpose of a transition probability matrix $M = (D^{-1}A)^T$ for algebraic PageRank computations. Create matrix M using the following digraph G: (10 pts)
   $V = \{v_1, v_2, v_3, v_4\}$
   $E = \{e_1(v_1, v_2), e_2(v_1, v_3), e_3(v_1, v_4), e_4(v_2, v_3), e_5(v_2, v_4), e_6(v_3, v_1), e_7(v_4, v_3), e_7(v_4, v_1)\}$

$$\underbrace{\begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}}_{D^{-1}A}$$

$$M = (D^{-1}A)^T = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

3. If we initialize PageRanks to be $\frac{1}{4}$ for all $v \in V(G)$, what are the PageRanks after a single iteration of computation using our linear algebraic model? (5 pts)

$$p_1 = Mp_0 = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/12 \\ 1/3 \\ 5/24 \end{bmatrix}$$

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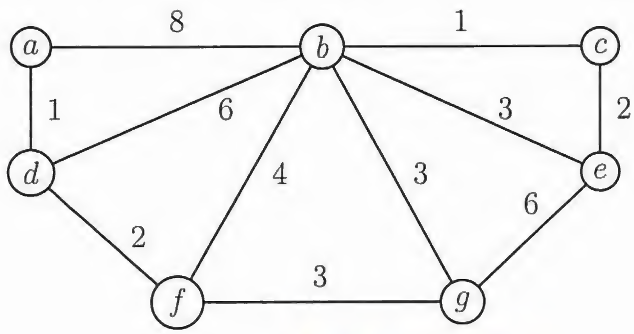
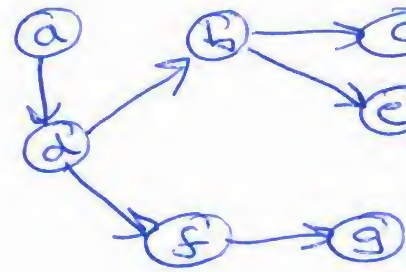
4. For the next two problems, consider the below graph. Using Djiktra's algorithm, calculate single-source shortest paths from vertex a to all other vertices. Also explicitly give the processing order of vertices during the algorithm. (10 pts)



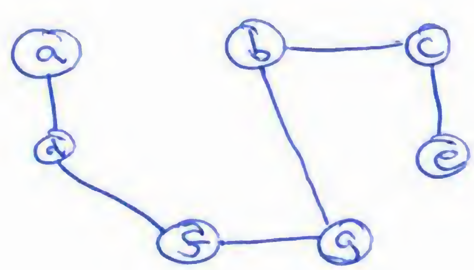


Process order	distances						
	a	b	c	d	e	f	g
–	0	∞	∞	∞	∞	∞	∞
a	0	8	∞	1	∞	∞	∞
d	0	7	∞	1	∞	3	∞
f	0	7	∞	1	∞	3	6
g	0	7	∞	1	12	3	6
b	0	7	8	1	10	3	6
c	0	7	8	1	10	3	6

e last, no changes

5. Give a possible order of edges added to a minimum spanning tree using Krushkal's algorithm on the graph given above. (10 pts)

(b, c)
(a, d)
(d, f)
(c, e)
(f, g)
(b, g)



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6. Bipartite graph $G_{X,Y}$ has a maximal, but not necessarily maximum, current match M, where $|M| = 6$ edges and $|X| = 7, |Y| = 8$ vertices. There exists at least one set $S = \{x, y, z\} : x, y, z \in X$ such that $|N(S)| > |S|$ and exactly one vertex of $\{x, y, z\}$ is unmatched, although no M-augmenting paths currently exist on G. Does G have a match M' that fully saturates X? Justify your response. (10 pts)

$\boxed{\text{No.}}$ By Berge's theorem the match is maximum as there exist no M-augmenting paths on G.

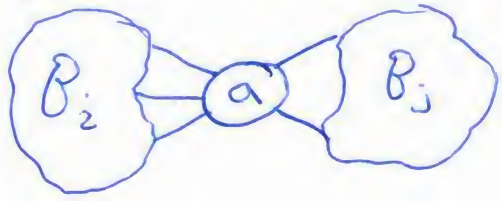
7. Connected graph G can be drawn as a block-cutpoint graph where no blocks are comprised solely of K_1 or K_2 and there is a nonzero number of blocks and articulation vertices. G has minimum degree $\delta(G) = 3$ and maximum degree $\Delta(G) = 5$. Give tight upper and lower bounds on G's vertex connectivity $\kappa(G)$ and edge connectivity $\kappa'(G)$. Justify your response. (10 pts)

G is connected => $K(G) \geq 1$

G has at least one articulation point/cut vertex => $\boxed{K(G) = 1}$

No K_2 blocks so no cut edges as each block is 2-edge-connected

However, as $\Delta(G) = 5$, each articulation point is at most 2-edge-connected to one block => $\boxed{K'(G) = 2}$



8. Consider a biconnectivity decomposition of G and its block-cutpoint graph. Prove that G is bipartite if and only if every block is bipartite. (15 pts)

G is bipartite => every block is bipartite

As every block is a subgraph of G, it trivially follows that they too must be bipartite.

Every block is bipartite => G is bipartite

Consider each block individually, do joining blocks through a cut vertex introduce an odd cycle? As a block-cutpoint graph is a tree, and each block cannot have any cycle that extends outside of it, there is no way to have some odd cycle introduced onto G. As no odd cycles <=> G is bipartite, then => G is bipartite □

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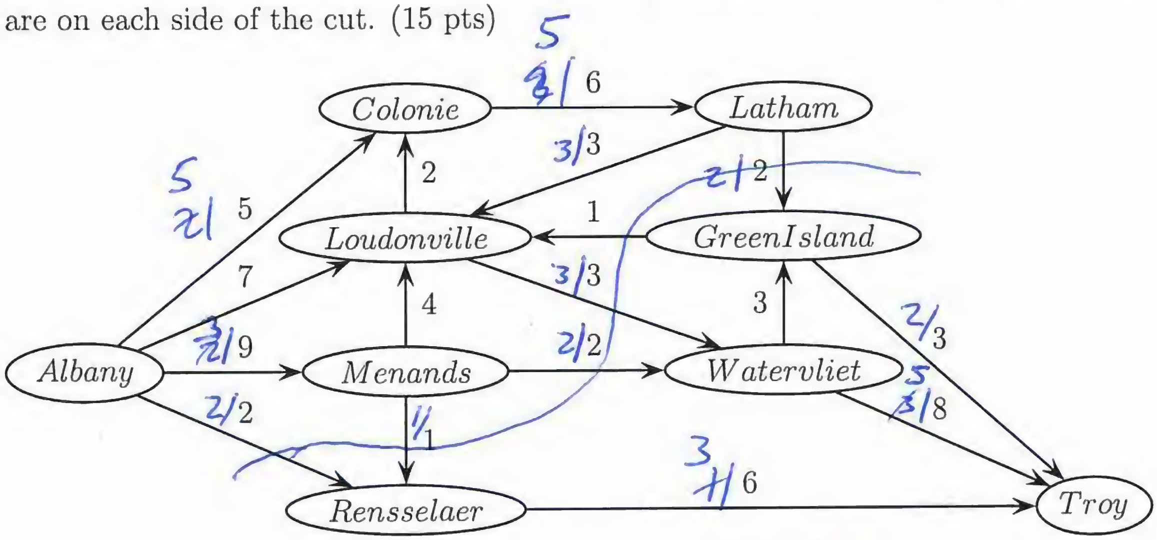
9. The US Census Bureau wishes to redraw the metropolitan boundaries between Troy and Albany. To do so, they consider a flow network of highway traffic between source vertex *Albany* to sink vertex *Troy*, where the minimum cut on this network will be the new boundary. To assist the Census Bureau, first calculate a maximum flow, use that to determine a minimum *Albany*− *Troy* cut, and then identify which towns are on each side of the cut. (15 pts)



Max flow = 3 + 5 + 2 = $\boxed{10}$

Min cut = max flow = $\boxed{10}$

Albany	Troy
Colonie	Green Island
Latham	Watervliet
Loudonville	Rensselaer
Menands	